# Simple ambiskew polynomial rings

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# skew polynomial rings

base ring $A$
$A[x;\alpha,\delta]$
$\alpha$ is a ring endomorphism of $A$.
$\delta$ is a (left) $\alpha$-derivation of $A$:

$$\delta(ab) = \alpha(a)\delta(b) + \delta(a)b.$$

As a group under $+$, $A[x;\alpha,\delta] = A[x]$ but

$$xr = \alpha(r)x + \delta(r).$$

Special cases: $\alpha = \mathrm{id}_A$ ($A[x;\delta]$); $\delta = 0$ ($A[x;\alpha]$).
From now on $k$ denotes a field.

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# Simple $A[x;\delta]$

$A$ is $\delta$-simple if there is no non-zero proper ideal $I$ of $A$ with $\delta(I) \subseteq I$.

$\delta$ is inner if $\exists a \in A$ such that $\delta(r) = ar - ra \; \forall \; r \in A$, outer otherwise.

Theorem: Let $A$ be a $k$-algebra, where char $k = 0$. $A[x;\delta]$ is simple iff $A$ is $\delta$-simple and $\delta$ is outer.

Examples

- $A = k[y]$, $\delta = d/dy$, $xy - yx = 1$ (the first Weyl algebra).
- $A = k[y, z]$: there are examples of $\delta$ with $A$ $\delta$-simple, due to many people, most notably Bergman and Coutinho.

There exists a more complex simplicity criterion in characteristic $p$.

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# Discussion point

What is the easiest example of a simple skew polynomial ring of the form $R[x; \alpha, \delta]$ with $\alpha$ outer?

There is an example due to Cozzens, reproduced over 5 pages of the 1975 book by Faith and Cozzens, of a simple ring $D[x; \alpha, \delta]$ where $D$ is a division ring and $\alpha$ is injective but not surjective.

There are known simplicity results due to Lam, Leroy $\pm$ Jain when $R$ is a division ring and to Lam, Leroy, Leung, Matczuk when $R$ is simple.

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# ambiskew polynomial rings

The data: a $k$-algebra $A$,
two commuting $k$-automorphisms of $A$, $\alpha$ and $\gamma$, $\beta := \alpha^{-1}\gamma$,
$v \in A$ such that $va = \gamma(a)v \ \forall\ a \in A$ and $\gamma(v) = v$ ($v$ is $\gamma$-normal),
$\rho \in k \backslash \{0\}$.
Form $A[y;\alpha]$ and set $\beta(y) = \rho y$ so that $\beta \in \mathrm{Aut}_k(A[y;\alpha])$.
The ambiskew polynomial ring $R = R(A, \alpha, v, \rho)$ is the iterated skew polynomial ring $A[y;\alpha][x;\beta,\delta]$, where $\delta(A) = 0$ and $\delta(y) = v$.

$$\begin{aligned} ya &= \alpha(a)y \quad \text{for all } a \in A, \\ xa &= \beta(a)x \quad \text{for all } a \in A \text{ and} \\ xy &= \rho yx + v. \end{aligned}$$

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$A = k, \alpha = \gamma = \mathrm{id}$

- $A_1(k)$, the first Weyl algebra: $xy - yx = 1$, $\rho = v = 1$.
- the quantum plane: $xy = qyx$, $\rho = q, v = 0$.
- the quantised Weyl algebra $A_1^q(k)$: $xy - qyx = 1$, $\rho = q, v = 1$.

$A = k[h]$: $U(sl_2)$,particular examples due to Witten, Woronowicz, Podleś, Kirkman/Small, general classes due to Smith, Rueda, Le Bruyn, Benkart/Roby (down-up algebras), Cassidy/Shelton (generalized down-up algebras).

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# Examples

$A = k[K^{\pm 1}]$: $U_q(sl_2)$

$A = k[b, c]$: $O_q(M_2(k))$

very recent and with combinatorial motivation
$A = k[h, K^{\pm 1}]$: augmented down-up algebras
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By iteration, higher quantized Weyl algebras.

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# Casimir elements

Suppose there exists a $\gamma$-normal element $u \in A$ such that $v = u - \rho\alpha(u)$.

$z := xy - u = \rho(yx - \alpha(u))$ is such that $zy = \rho yz$, $zx = \rho^{-1}xz$, $za = \gamma(a)z$ for all $a \in A$ and $zu = uz$.

Hence $zR = Rz$ and $R$ is not simple.

If such a $u$ exists then it is a splitting element, $z$ is a Casimir element and $R$ is conformal; otherwise $R$ is singular.

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# The elements $v^{(m)}$

For $m \geq 1$, $v^{(m)} := \sum_{l=0}^{m-1} \rho^l \alpha^l(v)$.
In particular $v^{(1)} = v$. $v^{(0)} := 1$.
Each $v^{(m)}$ is $\gamma$-normal.
For $m \geq 1$,

$$\begin{aligned} xy^m - \rho^m y^m x &= v^{(m)} y^{m-1} \quad \text{and} \\ x^m y - \rho^m y x^m &= x^{m-1} v^{(m)}. \end{aligned}$$

In the conformal case, $v^{(m)} = u - \rho^m \alpha^m(u)$.

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# Simplicity criterion for $R$

Theorem: Suppose that char $k = 0$. The ring $R$ is simple if and only if

1. $A$ is $\alpha$-simple;
2. $R$ is singular;
3. for all $m \geq 1$, $v^{(m)}$ is a unit of $A$.

There is a more complex simplicity criterion in characteristic $p$.

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# Examples of $\alpha$-simplicity

- $A$ a field extension of $k$ e.g. $k = \mathbb{R}$, $A = \mathbb{C}$, $\alpha(z) = \overline{z}$.
- $A = k[t]$, $\alpha(t) = t + 1$, char $k = 0$.In this case a splitting element $u$ always exists, of degree $1 + \deg v$ if $\rho = 1$ and $\deg v$ if $\rho \neq 1$.
- $A = k[t^{\pm 1}]$, $\alpha(t) = qt$, $q \in k^*$ not a root of unity.
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# Examples

If char $k = 0$, $A_1(k)$ is, of course, simple: $v^{(m)} = m$, $u - \alpha(u) = 0$.

$A_1^q(k)$ is not simple when $q \neq 1$: $u = 1/(1-q)$, $u - q\alpha(u) = 1$.

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# Examples

Consider $\mathbb{C}$ as an $\mathbb{R}$-algebra with $\alpha(z) = \overline{z}$ and $\gamma = \mathrm{id}_{\mathbb{C}}$. Taking $v = a + ib$ we get the $\mathbb{R}$-algebra $R$ generated by $i, x$ and $y$ subject to the relations

$$i^2 = -1, \quad xi = -ix, \quad yi = -iy, \quad xy - \rho yx = a + ib.$$

$R$ is singular if and only if either $\rho = 1$ and $a \neq 0$ or $\rho = -1$ and $b \neq 0$.
By the condition on $v^{(m)}$, $R$ is simple if and only if either $\rho = 1$ and $a \neq 0$ or $\rho = -1$ and $b \neq 0$.
If $v = 1 = \rho$, where $xy - yx = 1$, then $v^{(m)} = m$ and if $v = i$ and $\rho = -1$, where $xy + yx = i$, then $v^{(m)} = im$.

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# Example with $A = k[t^{\pm 1}]$

If $q \in k^*$ is not a root of unity and $0 \neq n \in \mathbb{Z}$ then the $k$-algebra generated by $t^{\pm 1}, x$ and $y$ subject to the relations

$$\begin{aligned} yt &= qty, \\ xt &= q^{-1}tx \\ xy - q^{-n}yx &= t^n. \end{aligned}$$

is simple.

# Example with $A = kG$, $G$ finite cyclic

Let char $k = 0$, $n \geq 2$ and $\varepsilon \in k$ be a primitive $n$th root of unity.

$R$: $k$-algebra generated by $t$, $x$, $y$ subject to

$$t^n = 1, \quad yt = \varepsilon ty, \quad xt = \varepsilon^{-1} tx,$$
$$xy - \rho yx = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}.$$

$R$ is generically simple. For example when $n = 2$, $\varepsilon = -1$ and $\rho = 1$ the relations become

$$t^2 = 1, \quad yt = -ty, \quad xt = -tx,$$
$$xy - yx = c_0 + c_1 t.$$

Here $R$ is simple if and only if $c_1 \neq 0$ and $c_0 \neq \pm m c_1$ for all odd $m \geq 1$.
You might recognise this as a symplectic reflection algebra.

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# Simple higher quantized Weyl algebras

Suppose char $k = 0$ and let $\Lambda = (\lambda_{i,j})$ be $n \times n$ such that $\lambda_{j,i} = \lambda_{i,j}^{-1}$ for $1 \leq i, j \leq n$. The $k$-algebra generated by $x_1, y_1, \ldots, x_n, y_n$ subject to the relations

$$\begin{aligned} y_j y_i &= \lambda_{j,i} y_i y_j, \quad 1 \leq i < j \leq n; \\ y_j x_i &= \lambda_{i,j} x_i y_j, \quad 1 \leq i < j \leq n; \\ x_j y_i &= \lambda_{i,j} y_i x_j, \quad 1 \leq i < j \leq n; \\ x_j x_i &= \lambda_{j,i} x_i x_j, \quad 1 \leq i < j \leq n; \\ x_j y_j - y_j x_j &= 1, \quad 1 \leq j \leq n, \end{aligned}$$

is simple. These are iterated ambiskew polynomial rings and special cases of a more general construction.

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# Removing the Casimir barrier

Suppose that $R$ is conformal so that the Casimir element $z$ is a barrier to simplicity. It can be removed by inverting $z = xy - u$ to form $S := R_{\mathcal{Z}} := R_{\{z^i\}}$ or factoring out $z$ to form $T := R/zR$. There are simplicity criteria for both $S$ and $T$.

# Simplicity criterion for $S$

Theorem: The ring $S = R_{\mathcal{Z}}$ is simple if and only if the following hold:

1. $A$ is $\{\alpha, \gamma\}$-simple;
2. there do not exist $c \in A$ and $m, j \in \mathbb{Z}$, with $m$ and $j$ not both 0, such that $\gamma(c) = \rho^m c, \alpha(c) = \rho^j c$ and $c\gamma^j(a) = \alpha^m(a)c \; \forall a \in A$;
3. for all $m \geq 1$, there exists $n$ such that $u^n \in v^{(m)}A$.

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# Examples

Examples with $S$ simple:

- the quantum torus $k[x^{\pm 1}, y^{\pm 1} : xy = qyx]$, $q$ is not a root of unity.
- $A_1^q$ when $xy - yx$ is inverted and $q$ is not a root of unity;
- localizations of non-simple higher quantized Weyl algebras;
- $A = k[h]$, $\alpha(h) = h + 1$, $\gamma = \text{id}$, $\rho$ not a root of unity,
- $A = k[h^{\pm 1}]$, $\alpha(h) = qh$, $\gamma = \text{id}$, $v = h$, $S$ is simple if and only if the subgroup of $k^*$ generated by $q$ and $\rho$ is free abelian of rank 2.

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# Factor out $z$

Abusing notation, in $T := R/zR$,

$$\begin{array}{rcl} ya & = & \alpha(a)y \text{ for all } a \in A, \\ xa & = & \beta(a)x \text{ for all } a \in A \text{ and} \\ xy = u & ; & yx = \alpha(u). \end{array}$$

This is a (generalized) generalized Weyl algebra in the sense of Bavula.

# Simplicity criterion for $T$

Theorem: $T$ is simple if and only if

1. $A$ is $\alpha$-simple;
2. $\alpha^m$ is outer for all $m \geq 1$;
3. $u$ is regular;
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